NTC : Quantum Pseudo telepathy

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A collaborative game :

Three persons are presented with the following game where they can at each round either be all eliminated from the game or win and continue playing. For each round of the game : they are placed in three different rooms and they are visited simultaneously by three examiners and each offered a single ball either black or white. They can either take it or let the examiner keep it.

They win the round if :

— They are presented one black ball (and two white balls) and they give back in total an odd number of balls.

— They are presented three black balls and they give back in total an even number of balls.

— They are presented an even number of black balls (no matter what they take).

Before the beginning of the round they can do what they want : they can communicate, prepare strategies, share information ...

1 Deterministic strategies :

We say that a ball is of color j, where $j = 0$ if it is white and $j = 1$ if it is black.

Let $a_i(j)$ be the number (0 or 1) of balls left in the hand of the examiner that visited the player i presenting to him a ball of the color j.

$a_i : \{0, 1\} \to \{0, 1\}$ is called a deterministic strategy for the player i.

Example $a_2(0) = 1$ and $a_2(1) = 0$ means that the second player when presented a white ball doesn't take it but when presented a black ball takes it.

- **Q1.1** Write 4 equations modulo 2 using the $a_i(j)$ (where $i \in \{1,2,3\}$ and $j \in \{0,1\}$) corresponding to the cases where the players win the round when they are (in total) presented an odd number of black balls.
- **Q1.2** Prove by contradiction that it is impossible for the players to be sure to never lose if each of them uses a deterministic strategy.

2 Probabilistic strategies :

Suppose that the players have access to common random variables : they can play probabilisitic strategies *i.e* with probabiliy p_k (that does not depend on the color of the balls presented to them but only on the random variable they all share) they can each apply a deterministic strategy depending on k (with $\sum p_k = 1$).

- **Q2.1** Suppose that the color of the balls presented to the players is chosen with a uniform independent distribution : for all the players and independently ($P(j=0) = P(j=1) = 1/2$). What is the probability of losing if the players chose to use the following probabilistic strategy : the identity with probability 1/3 *i.e.* with $p_0 = 1/3$, $\forall i, j$, $a_i(j) = j$ and the constant function 1 with probability 2/3 *i.e.* with $p_1 = 2/3$, $\forall i, j$, $a_i(j) = 1$?
- **Q2.2** Is it possible to ensure never losing with certainty using a probabilistic strategy ?

3 A Quantum strategy :

We want to prove that by sharing entangled states the players can be sure that they will never lose.

- **Q3.1** Write in the standard basis $\frac{|+\rangle|0\rangle+|-\rangle|1\rangle}{\sqrt{2}}$ (where $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$)
- **Q3.2** Write in the standard basis $\frac{|+\rangle|+\rangle+|-\rangle|-\rangle}{\sqrt{2}}$
  Consider the three qubit state
  $$|E\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle - |110\rangle - |011\rangle - |101\rangle - |111\rangle$$
- **Q3.3** Write the state $|E\rangle$ with the first qubit in the $\{+,-\}$ basis.
- **Q3.4** What are the possible results of the measurements if one qubit is measured in the $\{+,-\}$ basis (with $\{|+\rangle\langle+|, |-\rangle\langle-|\}$) and the others in the standard basis (with $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$).

- **Q3.5** What are the possible results of the measurements if the three qubits ard measured in the $\{+,-\}$ basis ?
- **Q3.6** Explain how the players can guarantee to always win the game. (Hint : consider the sum of the classical outcomes)

4 Preparing the state $|E\rangle$

We want to prepare the state $|E\rangle$

$$|E\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle - |110\rangle - |011\rangle - |101\rangle - |111\rangle$$

The CZ gate (Controlled Z) is the two qubit gate defined on the basis states by : $\forall a, b \in \{0,1\}$, $CZ\,|a,b\rangle = (-1)^{a.b}\,|a,b\rangle$.

- **Q4.1** What is the image by CZ of $|0\rangle\,|\varphi\rangle$ where $|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$?
- **Q4.2** What is the image by CZ of $|1\rangle\,|+\rangle$ and $|1\rangle\,|-\rangle$?
- **Q4.3** Draw a circuit representing CZ using $Hadamard$ and $CNot$ gates (recall that H maps $|0\rangle$ to $|+\rangle$ and $|1\rangle$ to $|-\rangle$ and $CNot$ maps the basis state $|a,b\rangle$ to $|a,a+b\rangle$ where the sum is modulo 2)
- **Q4.4** Write in the standard basis the state obtained by applying CZ to the first two qubits of the state $|+++\rangle$.
- **Q4.5** Draw a circuit that maps the state $|000\rangle$ to the state $|E\rangle$ (defined in the quantum strategy section) using $Hadamard$ and CZ gates.
- **Q4.6** What is the density matrix representing the state of the first qubit of $|E\rangle$?
- **Q 4.7** Find three unitaries on three qubits U_1, U_2, U_3 that are tensor product of X and Z (Pauli operators) such that $U_i\,|E\rangle = |E\rangle$ (these operators form the *stabilizer* of $|E\rangle$)